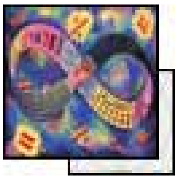
Infinity



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The term “infinity” has, since ancient times, been used by philosophers, theologians, mathematicians, and poets, which has resulted in its having a vast range of meanings and applications. The term is applied in an “objective” sense to indicate something that has the characteristic of being “unbound” (e.g., a straight line, the sequence of integer numbers, etc.), or may also be applied objectively to indicate the “absolute perfection” of a Being, essentially with reference to God. The notion of infinity is applied in a “subjective” sense to indicate the perception had by an experiencing subject who considers something to be infinite that, objectively, is finite but is much greater than the perceiving subject (the height of the sky, the breadth of the ocean, etc.). In this last case the term is used in a relational way to indicate a sort of “ratio” or “scale” between the subject and the object. This relationship can refer to the infinitely large or be inverted with respect to the infinitely small (the infinitesimal). Etymologically the significance of the term includes a negation characterized by a negative preposition (Gr. *à–peiron*; Lat. *in–finitus)*: non-finite, that is, without boundaries or limits (Lat. *fines*), not only in an “extensive” sense, but also in the “intensive” sense of limitations and imperfections, and therefore supplied with the fullness of positivity, that is to say of total “actuality.” Infinity, as extensively agreed upon, therefore presents itself as not completely crossable, not traversable from side to side (e.g., a straight line cannot be completely run through). As intensively agreed upon, infinity consequently presents itself as not totally graspable and as such is inexhaustible or incomprehensible (Lat. *in-comprehensus*, non-circumscribable), that is, unavailable to a complete intellectualization by means of specific acts of human knowledge, which are by their nature necessarily limited and finite in number. In its negative sense, indicating something that has no limits or boundaries, the term is sometimes employed as synonymous with “indefinite” and therefore as “indeterminate” and “amorphous”: It indicates that which is without properties and face and thereby totally deprived of any “actuality,” not as though it were nothing, but rather as a pure disposition to receive whatever determinations, that is, a pure potentiality. In the history of thought, the philosophical approach to infinity has been marked, particularly from Aristotle onwards, by the notion of "potential" infinity, while in the theological sphere Thomas Aquinas highlighted the property of "actual" infinity. In more recent times, Georg Cantor elucidated infinity in mathematics with his enrichment of the notion through the concept of the "transfinite."

I. Infinity: From the Ionian Philosophers to Aristotle

1. The Presocratic Philosophers: The Ionians, Pythagoras, Parmenides, and Democritus. The first philosophical school, the School of Miletus (Asia Minor), was characterized by seeking the principle (Gr. *arché*) of all things, which Thales (640-560 B.C.) considered to be water. For Anaximander (610-547 B.C.) all things are properly regarded as "definite," whereas their principle is considered "indefinite" (*á-peiron).* Thus, the term "infinite" appeared for the first time with the sense of "indefinite" and "indeterminate." For Anaximenes (585-528 B.C.), the materialization of the *ápeiron* of Anaximander brought him to believe air (through rarefaction and condensation) was the origin of all things.

In the school of Pythagoras, which developed in Croton in Greater Greece in the 6th century B.C., beginning with the teaching of its founder Pythagoras of Samo, the theses of Anaximander and Anaximenes were elaborated—for the first time in the history of Western thought—resorting to a mathematical foundation. For Pythagoras, all things derived from the synthesis of the "definite-indefinite" and the "limited-unlimited." The essence of all things is that which pertains to geometrical figures. They are ultimately formed by points or undividable unities and therefore by number-points. The things that are definite are so because they are "measurable" (since they are extended entities or geometrical figures) and "denumerable" (since they are composed of undividable unities). Reality, then, is born from the harmony of opposites and most importantly from the fundamental opposition of the "limited-unlimited," or respectively of the "uneven" (*one,* limited, form) and the "even" (*two,* unlimited, matter), since from one and two all numbers and all geometric shapes can be constructed. The other oppositions that derive from the preceding ones are those of "straight-curve," "rest-motion," etc. In this philosophical thought a glimpse is caught of the role of the unlimited, of infinity understood as a "disposition" to receive determinations.

An initial connection between the metaphysical and the anthropological contexts arose with Parmenides of Elea (520-440 B.C.) who for the first time in Western thought brought to light the notion of "being." Parmenides affirmed the identity between thought and being: He conceived of being univocally, like a "very general genus," as a unique notion, more universal than the others, and one not considered "according to diverse meanings," as Plato and Aristotle would discover and later set forth. Parmenides formulated that fundamental law of logic that is the "principle of non-contradiction," which for him was also the fundamental metaphysical law. But he did not recognize the distinction between a metalogical and a metaphysical use of that principle, which is indispensable to avoid confusions and errors. With this premise, Parmenides affirmed the purely apparent character of quantitative multiplicity, of qualitative diversity, and of becoming in its different forms. Melissus of Samo, his disciple, probed into the contradiction of this vision: If being is a unique entity, it cannot be unlimited, because "unlimited" indicates non-being, a negativity (here "infinite," as lacking delimitations, is meant as "indeterminate" and imperfect). It will therefore be limited, in the way that a "sphere" is limited. But if it is limited, and all being is by definition in the sphere, who limits the sphere? Such limits must be "outside" of the sphere. But if all being exists within the sphere, who or what could limit it except non-being? Yet, non-being does not exist and, whatever being may be considered, limited or unlimited, it will have to do with non-being: Herein lies the antinomy. As can be seen, Parmenidean metaphysics, like all other rationalistic metaphysics—and also, today, like any formal metalogic—tends towards antinomy.

Democritus of Abdera (460-370 B.C.) replied to Parmenides by demonstrating the non-contradiction of multiplicity. In fact, if Zeno (a disciple of Parmenides) had demonstrated the contradiction of the infinite divisibility of extended material (cf. Koyré, 1971, pp. 9-35), it would furthermore be necessary to admit the existence of utmost undividable parts (atoms), and thereby extended reality would need to overcome the accusation of the contradiction of the notion of numerical multiplicity pointed out by Parmenides and Zeno. Therefore, in order to justify the concept of multiplicity it is not necessary to invoke the existence of "nothing" (absolute non-being), because it is enough to employ the concept of "void" (i.e., relative non-being or the absence of matter). A void, as an entity, is not an absolute non-being but the pure and simple "deprivation" or "absence of matter." The contradiction arises from the "absolute" opposition between being and non-being, but nothing forbids that something is with respect to some particular thing, and is not when compared with something else. The void is the "absence of matter," not the absurd existence of non-being. The void is not nothing but rather the "non-being of something." Analogously, when related to the number "1," by which we enumerate a certain discreet or atomic entity, the number "0" is not completely representative of "nothing" but rather denotes the absence of that entity or, more exactly, the emptiness of it (the *empty set* as referred to by the mathematicians within the framework of modern set theory). But this last refinement entered mathematical thought later on, in the Middle Ages, with Arabic mathematics. All of this opens up the door a bit more to the possibility of considering an entity which is "infinite" in a positive sense, i.e., without the relative negation of being and no longer in an indeterminate and negative sense.

2. Plato and Aristotle. With Plato, and afterwards and above all with Aristotle, the explanation of the concept of being would be developed, no longer univocally, but taking into account that "being" is said in different ways, that is, the term being is applied analogously. An important change took place here from the purely negative notion of infinity (such as an indeterminate one) to one that is positive (infinity "in act"). Plato took the first step in this direction with his dualistic conception. For Plato, in the cosmos two worlds exist: One is material, composed of entities in a continuous state of becoming, and the other is "immaterial," composed of immobile entities in a fixed state, which are not becoming. This is nothing but the definition of "difference" that, even within the limits of a dualistic scheme, already implies the surmounting of the presumed contradiction affirmed by Parmenides, and therefore introducing the possibility for multiplicity. The "difference" between *A* and *B*, for example, although implying that *A* means *non-B* and that *B* means *non-A*, no longer implies the notion of an absolute non-being. That is to say, it denies only something determinate of *A* or of *B*: It does not deny "all of *A*" or "all of *B*," but only the "form *a*" of *A* and the "form *b*" of *B*. Difference, therefore, implies *relative* non-being, not *absolute* non-being. The "different" entities are not opposing each other due to an opposition of contradiction (*A/non-A*), but they are simply opposed due to the opposition of contrariety (*A/non-a*). It refers to an opposition regarding the form, not an opposition with respect to the whole being, because affirming *B* does not deny all of *A*, but rather only its *a* form. Evidently, then, the entity *A* or the entity *B* is not only composed of "form" but also of "matter," physical or intelligible, corresponding to the reference of a physical or logical entity. The development of this very valuable Platonic insight would be accomplished by Aristotle with his doctrine of act and potency.

Through the distinction of two irreducible principles, "matter" and "form," constitutive of the essence of every physical entity, regardless if it is a substance or an accident, Aristotle furnished a reply to the problem of Parmenides concerning the presumed contradiction within the concept of becoming. In order to realize this, Aristotle considered matter [2] as a kind of "potentiality to exist," or as "potential being," and form as the "actuality of being" that actualizes matter and determines it to exist as a specific actual being. "Becoming" is not a transition from being to non-being, or vice-versa, but rather the transition from the state of potential being to the state of actual being and vice-versa. The distinction between potential and actual being allowed Aristotle to introduce a diversification of the entities that account for the various "nuances" encountered in the experience of real beings. It also allowed him to consider infinity in terms of potency and act. In this way the distinction was born, which later became classical, between "potential infinity" (always subject to "enlargement" and never considered within its totality) and "actual infinity" (which is a concept of infinity with respect to fullness, considered simultaneously and in its totality).

The potential infinity of the "matter" introduced by Aristotle is considered an essential, irreducible indetermination of the finite, material substratum of a physical entity. This indetermination of infinity is defined by Aristotle (cfr. *Physics*, III, 206a) in a way similar to that which would later be introduced by Cantor (see below, IV). The potential infinity of matter is what matter without form would be, an "ever different being," the pure causality of becoming. For example, a casual sequence of numbers is a sequence in which no repetition can be found, no periodicity, no function or "law" capable of predicting successive values on the basis of the preceding data. Matter [2], as a constitutive principle of each physical entity, refers essentially to the intrinsic instability of the motions of a substratum of elements. Therefore, according to Aristotle, matter deals with both potential being (matter itself) and actual being (matter that receives a form). The potential infinity of physical matter is, in short, an absolutely unpredictable becoming, without any stability or periodicity, and without any law or mutual ordering of the parts, neither in space nor time. For Aristotle, then, physics cannot be reduced to geometry, contrary to the majority of Greek and modern thought. It is worth highlighting how Aristotle's philosophy of nature seems to be the philosophy which gives the best basis for state of the art contemporary physics since the discovery of the prevailing role of dynamic instability in the study of real physical systems.

II. The Reflections of Thomas Aquinas and the Theological Significance of the Notion of Infinity

With respect to Aristotle, Thomas Aquinas (1224-1274) pointed out some important aspects of the problem of infinity that have in more recent times been confirmed by the mathematical approach to infinity as it was developed by Georg Cantor and which is today considered to be classical. Aquinas demonstrated that while the notion of "infinity *in act*" is, as such, contradictory, it is not contradictory, instead, to refer to "*actually* infinite objects," both "absolutely" and "relatively." Regarding this, it is of fundamental importance to emphasize that, unlike what is commonly believed, Thomas did not reject actual infinity as such, but he specified the conditions that are necessary to correctly refer to it. In particular, it is important to understand in which sense Aquinas considers it contradictory to speak of "collection-objects infinite in act" (*infiniti in actu*), while it is not completely contradictory to speak of "*actually* infinite objects" (*actu infiniti*), both relatively (*secundum quid*), and absolutely (*simpliciter*). Bearing in mind that within the context of actually infinite objects (in an absolute sense) there can exist only one Absolute or Subsistent Being—which are philosophical attributes Aquinas recognizes as proper only to God [3] —the medieval philosopher proposes a distinction regarding three kinds (Lat. *genera*) of infinity: a) objects that are "potentially infinite"; b) objects that are "actually infinite in a relative sense"; c) objects that are "actually infinite in an absolute sense." These are all perfectly consistent as opposed to the contradictory notion of "objects infinite *in act*." The Thomistic conception corrects and enlarges that of Aristotle who recognized only two types of infinity within science, namely, that regarding potency (consistent infinity) and that with respect to act (inconsistent infinity). It would therefore be erroneous to maintain that the adjustment carried out by Cantor—using the methods of set theory to rigorously confront the different types of infinity that exist within mathematics—rejects the Scholastic philosophical doctrine that considered the notion of infinity in act contradictory, except when applied to divinity.

A careful study of Aquinas' work demonstrates that the notion of "relative actual infinity" (*infinitum actu secundum quid*) was perfectly recognized by him to be non-contradictory and therefore admissible as a logical entity, though not as a "constructive" notion (that is, as *infinitum* "*in actu*"), because it would be contradictory to conceive of an infinity as potential and, at the same time, as completely actualized. It therefore makes sense for Aquinas to speak of the actuality of a "relative infinite" only as the negatively conceived infinite of a certain totality, according to one of its specific modalities of being. For example, Aquinas says that it is perfectly consistent to affirm that the totality of the natural numbers is infinite because there does not exist, nor *can* there exist, a maximum natural number of each infinite sequence of natural numbers. In other words, if one wishes to speak of an "upper bound" of a set such as the natural numbers, one must consider it as "exterior" to the set of the naturals; it must be thought of as something belonging to another *kind* (Lat. *genus*)—a transfinite number in the sense of Cantor—and not belonging to the sequence itself as its "maximum" number.

Regarding infinity considered as an attribute of God, one should keep in mind that St. Thomas dedicated a whole question to the subject in the First Part of the *Summa Theologiae* (Part I, q. 7, "The Infinity of God") immediately following the question regarding the "Goodness of God" and immediately prior to his exposition about the "Presence of God in All Things" (in which he gathered together some of the results of his previous reflections on infinity, see q. 8, a. 4). In that question, God is recognized as the "infinite and perfect Subsistent Being." The order of the four articles descends from the philosophical-theological level to the physical level and finally to the mathematical level: He passes from the question "Whether God is infinite?" (a. 1), to "Whether there is anything other than God which is essentially infinite?" (a. 2), and "Whether anything can be actually infinite in size?" (a. 3), finally asking "Whether there can exist an infinite number of things?" (a. 4).

Within the logic of this question, the infinity of God derives from the reality of God being Pure Act, that is, from the convergence between His fullness of Being and His unconditional capacity to be *causa omnium formarum*, without having within Himself any composition of matter. When attributed to "matter," infinity always contains some restriction, but "form," on the other hand, is *itself* unlimited and only knows limitation if it is joined to the matter it actualizes, something that cannot happen in God. The infinity that is attributed to God is not the infinity that refers to quantity but rather that which refers to Pure Act. This Act determines, and therefore transcends, every quantity in the material order. The infinity of all the forms destined to be joined to matter (and also the infinity of matter destined to be joined to many possible forms, for example, if one considers the various forms which the matter of a specific piece of wood could assume as it is fashioned by an artist) is always limited (and therefore only infinite in a relative sense) because it is conditioned by the "relative" infinity of that which is a composition. Regarding the angels, in whom there is no composition of form with matter, the non-absolute infinitude of their forms is due to the fact that they possess, even in their spiritual multiplicity, a "determined" nature, something that cannot be said of God. If something other than God is infinite, it could only be infinite in "a relative sense and not in the full, absolute sense" (q. 7, a. 2, resp.). It follows that there cannot be absolute actual infinities in the dimensional order, i.e., bodies (physical or geometric), because they are entities that always possess a determined form.

The theme of God having the attribute "Infinite" remains dear to medieval theology, which gives what is perhaps the first speculatively interesting formulation in the reflections of Anselm of Canterbury (1033-1109), when at the beginning of the *Monologium* he refers to God as "that than which nothing greater can be thought" (*id quod maius cogitari nequit*). Later on, in the itinerary of the *Proslogium,* Anselm speaks of God as "something always greater than that which can be thought" (*quiddam maius quam cogitari possit*). The first case addresses a notion that any human being is by nature predisposed to appreciate the meaning of, while the second case is rather a conclusion reached by a thinking believer (in this case, Anselm) who compares his knowledge of God with everything that is not God. Akin to other attributes such as Omnipotence, Truth, Goodness, or the Omniscience of God, the philosophical attribute of God's "Infinity" is found in some Roman Catholic professions of faith such as that enunciated by the First Vatican Council (1870) [4]: "Creator and Lord of heaven and earth, omnipotent, eternal, immense, incomprehensible, infinite in intellect and will, and in every perfection" (*DH* 3001). On the other hand, it must be kept in mind that the term is predominantly philosophical and as such it is hardly present in Scripture. In fact, Scripture considers the theme of divine "infinity" in ways which are mainly existential and historical-salvific in character such as God's omnipotence and lordship over history and His incomparable love for humankind. This has been revealed to us in Christ, as beautifully expressed in the Pauline letter: "And that Christ may dwell in your hearts through faith; that you, rooted and grounded in love, may have strength to comprehend with all the holy ones what is the breadth and length and height and depth, and to know the love of Christ that surpasses knowledge, so that you may be filled with all the fullness of God" (*Eph* 3:17-19).

III. Infinity within Galilean and Newtonian Science

The birth of modern science and mathematics, each discipline reciprocally enriching the other, occurred with Galileo Galilei (1564-1642), particularly with the proof he offered for the law of falling bodies and the consequent necessity to admit "actual infinity" as an indispensable component of the theoretical framework of the "new" science (cf. Koyré, 1957). In fact, the Galilean supposition that each moving body must actually go through "all" the infinite "*momenta* of smaller fastness (or greater slowness)," contrary to the medieval (non-Aristotilean) theory of impetus, necessarily implied the acceptance of the actuality of infinity. Torricelli (1608-1647) and Cavalieri (1598-1647)—both of whom followed Galileo, applying his "geometry of infinitesimals" though they had no knowledge of Archimedes' *Metodo* which was discovered in 1906—employed the idea of the divisibility of a continuum within an actually infinite set of undividable parts. Cavalieri, as opposed to Toricelli who was more pragmatic, thought that one could ascribe to the method of the indivisible infinitesimals (Cavalieri's Principle) the dignity of a formal demonstration, not considering it a simply heuristic procedure, as Archimedes supposed. Cavalieri maintained this regardless of the fact that he never affirmed explicitly that the continuum was composed of an infinity of indivisible infinitesimals, an assertion that is not rationally demonstrable (cf. Koyré, 1973, pp. 334-361).

Galileo, rather, introduced a principle in all his scientific discussions that I could call "Parmenidean." He linked the existence of infinitesimal elements, on the one hand, to the *necessity* to conceive them, and on the other hand to the non-contradictory nature of their existence, following in this respect a very particular way of reasoning. He affirmed that the continuum is composed of the infinite indivisible "non-quanta" it possesses (i.e., unextended elements). Thus he intended to overturn the criticism of the Aristotelians affirming that, if an extended quantity can be divided into an infinite number of parts, then it is supposed that the parts themselves are infinite, otherwise the "division would stop" at some step. However, if these infinite parts were "quanta" (extended), their infinity would imply that the resultant extension of their composition was infinite. Therefore every continuum is composed of infinite undividable parts that are non-quanta (cf. Lombardo-Radice, 1981, p. 33).

To understand a similar, rather original, way of reasoning we must consider that for Galileo, in compliance with Greek Euclidean mathematics, the arithmetical "unities" were not "numbers" but rather the expression of proportionality ratios existing between continuum quantities. This principle had an immediate experimental significance for him, since he carried out his calculations through the use of only integer numbers and fractions: These were ratios derived from the measurement of continuum quantities, which he subsequently varied in order to render all such ratios “rational.” As historians have demonstrated, he worked with two parallel tables, one which reported the variations (double) observed for the length *l* of his pendulum, and the other that reported the times *t* necessary for the pendulum to complete a half period of oscillation, measured in “grains of water” by his very ingenious water clock (by which he could regulate the intensity of the water flow). Employing such parallel lists, he discovered the law of falling bodies in the form of the square of the times. In fact, he realized that, assuming each time *t* was the geometric mean between the number 2 and the length of the pendulum, the two tables become correlated row for row. Properly varying the two magnitudes in question, he further verified, in short, that the relation was biunivocal only if this relationship was satisfied (cf. Drake, 1990).

When considered within its original experimental context, one can better understand Galileo’s idea of the continuum as composed of indivisible non-quanta. He attempted to lead the new experimental science back to the great Greek tradition, which did not consider unity (a fundamental of discreet quantification) a number, but rather a generator of reciprocally irreducible numerical sets, obtained through the sequence operation. On the other hand, the beginning of modern mathematics can be identified precisely by the renunciation of this reciprocal irreducibility between numerical sets founded on the non-numerical character of unity; modern mathematics considered as numbers those irrational limiting of ratios that the Greeks considered to be non-numbers. This consideration, through the invention of infinitesimal calculus and the progressive definition of the concept of limit, brought us to the theory of real numbers as conceived by Dedekind, and therefore to the Cantorian theory of sets, which was the first attempt to give a coherent systematics to the whole problem. Correlatively, the history of modern science brought about the renunciation of the original formulation of infinitesimal calculus based on “indivisible infinitesimals” defended by G. Leibniz (1646-1716), thus resolving the paradox of Zeno, implicit in the affirmation that the sum of the infinite quanta parts brings us always to an infinite result, thanks to the discovery of convergent series and their application to the physics created by I. Newton (1642-1727). This discovery, which was closely related to that algebraic formulation of geometry of R. Descartes (1596-1650), was a significant contribution. To found infinitesimal calculus, it seemed the concept of actual infinity was no longer needed, or alternatively there was no longer need to consider “all” the points, “all” the lines, or “all” the planes included between two points on a straight line, between two sides of a polygon, or between two surfaces in a volume. Starting instead from a “finite” subdivision, an approximate calculus can be made to reach a limiting result. The approximated calculus is then transformed into an exact calculus when the number of the parts tends toward infinity, every part becoming “evanescent,” and in this sense infinitely small, that is to say “infinitesimal.” Thus the notion of “limit” was implicitly introduced by Newton, a notion which through successive stages, became formally rigorous in the 18th century. This led finally to the days of Georg Cantor (1845-1918).

IV. The Infinity of Cantor

1. Three Kinds of Infinity. Cantor considered three kinds of infinity (cf. Hallett, 1984, pp. 8-10) with a modality of distinction that has many points in common with Thomistic conceptions (see above, II): a) “potential” infinity, indeterminate, and capable of incremental increase; b) “transfinite” infinity, determinate, and capable of incremental increase; c) “absolute” infinity, determinate, and incapable of incremental increase. Also, for Cantor the latter infinity can be said only in reference to God, the Absolute Being. He made this distinction in order to oppose two kinds of ideologies typical of the Enlightenment regarding modern science and mathematics (one of a Spinozian nature, the other Kantian) and to refute their two fundamental principles.

The former is based on the Spinozian equivalence between God and Nature (*Deus sive Natura*), which is a fundamental principle of modern theoretical pantheism [5]. Such equivalence is based upon the supposed reducibility of both notions (God and Nature) to the notion of a completely determined actual infinity, which is taken as the immanent foundation for both the order of nature itself, and for the necessary and universal character of the explanations of the "new geometrically conceived Galilean-Cartesian science" of nature. The latter is the Kantian conception that considers actual infinity a limit towards which potential infinity tends and is founded upon the four Kantian anti-metaphysical antinomies regarding the idea of the world: finiteness/infinity, discretion/continuity, indeterminism/determinism [6], caused/uncaused. All of these antinomies are based on the conception of absolute actual infinity understood as the limit towards which the finite tends.

Against both of these bastions of rationalist 18th century philosophy and its anti-metaphysical program, Cantor poses the distinction between relative actual infinity, or the "transfinite" as a mathematical notion, and "absolute" actual infinity as a metaphysical and theological notion, typically attributed to the divine nature, and absolutely unreachable through purely mathematical knowledge. Unfortunately, Cantor thought his view on infinity opposed the Thomistic conception due to Cantor's incomplete knowledge of Aquinas' thought, together with the insufficient scholarship of some of Aquinas' followers. Therefore he was led to believe that he had to systematically oppose the Scholastic philosophical doctrine with his own conception of actual infinity in mathematics. The necessity of actual infinity reappears here in a sense that joins the "Parmenidian" instance with the "Platonic" instance. The necessity for the existence of actual infinity is linked by Cantor with the necessity for its conceivability (Parmenidian instance), precisely in relation to the rigorous definition of the notion of limits within analytical calculus and regarding the definition of Dedekind of a "real number" as the limit of a sequence of "rational numbers" not belonging to the sequence itself. These two notions in fact imply that, in order to let mathematics be founded on them in a truly self-consistent way (Parmenidian instance), the indefinite variation of the finite (potential infinity) requested by the notion of limit has to suppose the a priori "complete determination" of the domain of variation (Platonic instance). "There is no doubt that we cannot do without the *variable* quantities within the sense of potential infinity; and that from this can be demonstrated the necessity for actual infinity. In order that there is a variable quantity in a mathematical theory, the "domain" of its variability must be, strictly speaking, known ahead of time through a definition. Thus, the said domain must not be itself something variable, otherwise every base founded for the study of mathematics would vanish. Consequently, this domain is a definite set of values, and is thereby actually infinite" (Cantor, 1886, p. 9).

The three notions of infinity are strictly related to the three fundamental principles that give unity to all Cantorian thought: reductionism, finitism, and the instance (or the postulation) of an Absolute. The necessity that mathematics, at its more fundamental level, must deal with "complete domains of variation," considered as its proper object, brought Cantor to affirm his thesis that, at such a foundational level, mathematics must concern itself with collection-objects, or to be more precise, with "sets." The operation of reducing to sets all the objects which are conceivable without contradictions can be defined as the "reductionism [7]" of the set theory of Cantor. He decided afterwards to regard the same mathematical infinity as "finite objects," that is to say, as "closed totalities," a concept that brought him, as we know, to develop his theory of the "transfinite." He hoped to be able to reduce within the notion of the transfinite the concept of the continuum, in particular that which is linked to the re-elaboration set forth by Dedekind. Finally, and probably as a result of a deep-seated Platonic prejudice, he perceived the necessity that all conceivable mathematical entities, and therefore all of the sets, were parts of some "absolute collection," in which they would all be defined, in order that they could enjoy utmost consistency. At the same time, it was necessary that this absolute collection was not mathematically determinable, and in particular that it was not itself a set, in order to escape from the corresponding antinomies.

Cantor’s genius, universally recognized in mathematics only after David Hilbert (1862-1943), consisted in the fact that he understood the nucleus of every antinomy. Though he was close to various aspects of the thought of Von Neumann (1903-1957), Cantor distanced himself from him because the foundation Cantor gave for the absolute collections was not axiomatic but rather metaphysical. However, the Cantorian demonstration, through the antinomy of the “power set” (the set of all subsets of a given set), and the inconsistency and contradictory nature of the “universal set,” or “set of all sets,” is not a sort of accident within the course of the development of his research. It is something profoundly linked to the Platonic roots of his metaphysics regarding the ultimate non-predicability of the One that, at least in this context, is profoundly anti-Parmenidean. The Absolute cannot be conceived, without contradiction, as a univocally definable specific set, or more precisely, as the “totality of being” or as a “semantic whole.” At the same time, the other root of the Cantorian metaphysic (that is, the Parmenidean one) linked to the assumption that conceivability formally implies existence, helps us to see that, indeed, one cannot be Platonic without being Parmenidean. In Cantorian terms: “That which is not logically inconsistent exists” and, further, it exists within the “absolute collection.” It follows that each entity exists insofar as it belongs to the said Absolute collection, and vice-versa, each entity belonging to the Absolute collection is implicitly (unpredicatively) considered as actually existing. From this point arises the authentic “original structure” of every contradiction.

This ruinous “indecisiveness” between the Platonic instance and the Parmenidean instance is, in my opinion, the ultimate root of the theoretical weakness of the Cantorian theory of sets. The first implies the non-definability of the Absolute as a set or a “closed” infinite totality, and therefore its “transcendence” with respect to predicative mathematics and its simultaneous necessity with respect to mathematics as Absolute collection where all the non-inconsistent entities already exist and remain for ever. The second instance implies that existence depends only on logical non-inconsistency and therefore on belonging to the universal set. The question about the “unity” of the concept of set, and about the foundation for this unity, is the central question and the culmination of the Cantorian reflections, particularly after the discovery of the antinomy of Burali-Forti (1861-1931) which points out the inconsistency of the idea of “absolute ordering.” This consideration, which arose from within Cantorian theory, prompted Cantor himself to reflect on the real consistency of his idea of absolute collection without, however, leading him to doubt the existence of a set that ultimately must not be founded on its state of belonging to the absolute collection.

After setting forth the philosophical problem of correctly understanding the Cantorian concept of “actual infinity,” the next step is to consider the more constructive aspect of his set theory, i.e., his positive consideration of actual infinity, the “transfinite.” This notion immediately introduces the consideration of the antinomy of the “power set.” According to Cantor, this antinomy is a very valuable verification that the Absolute cannot be conceived of as an ordinary set.

2. The Notion of Transfinite. In his study of infinity within the fields of mathematics and philosophy, when introducing the theory of Cantor’s concept of the transfinite, Lucio Lombardo Radice (1981) affirms that the constructive center of the idea of Cantor is the criticism of the notion of limit that Kant had introduced into modern philosophy, along with his antinomies of the continuum. That is to say, it is the criticism of the belief that “the *limiting* entity of the finite is the *Absolute*.” This would be true if only one mode of considering actual infinity existed, but for Cantor this is not so. The transfinite infinity also exists, or to be more precise, the “determinate” infinity, specified within a limit, yet nevertheless “capable of incremental increase,” where not all the terms of the series are “actually defined” (i.e., enumerated). Cantor showed that every infinite “countable” set of elements (by which he means a set) that can be put in a biunivocal correspondence with the infinite denumerable set *N* of the naturals (*N* = {1,2,3,4, …, *n*, …}) "has the same infinite cardinal number of elements." It is easy to show that, for example, the set *Z* of the relative integer numbers (*Z* = {… –n … -3, -2, -1, 0, 1, 2, 3, …, *n*, …}) is countable. With an analogous procedure, it can be also shown that the set *Q* of all the rational numbers is countable and that even the set *U* of all the countable sets is, again, countable. This very powerful theorem demonstrates that an infinite set can have the same "power" (cardinality) as one of its proper parts (subsets). But to affirm that an infinite set has the same power as one of its parts does not signify an absolute negation of the classical principle according to which "a part cannot be equal to the whole." As a matter of fact, "having the same power" doesn't absolutely mean "being identical" (cf. Lombardo Radice, 1981, pp. 52f). A "part," by definition, lacks something that the "whole" has and therefore can never be identical to the whole. For the same reason, much attention must be given to the attribute "actual infinity" applied by Cantor to his concept of the transfinite. In the language of Thomas Aquinas, the infinities of Cantor's theory that are "determined but capable of incremental increase" (that is, the infinite totalities that he teaches us to treat actually) would simply be *"actual" infinities which are not "in act,"* since they are capable of incremental increase. That is to say, they are infinities only under some "special respect" (*secundum quid*), while they are essentially "finites." This is because all that is determined and/or specified is for this same reason "delimited" within an essence, even if it is "infinite" with respect to some modality or property. In the case of Cantor, the element of indetermination of the transfinite is really the fact that it is capable of incremental increase, the fact that an infinite totality can be conceived as specified in itself, even if its elements cannot all be caught simultaneously in detail. "The terms of an infinite series are all given by the constructive law provided to define the series, which makes it useless to enumerate them individually: It is the law that characterizes the series, and no longer 'all' the terms actually enumerated" (Zellini, 1980, p. 195).

Starting from the specific difference of "enumerability," Cantor defined a criterion of uniformity that allowed him to rigorously extend the concept of denumerability to numerical sets other than *N*. Having discovered how to correlate the elements of an infinite set in order to show if it is countable (to put their elements into a biunivocal correspondence with those of *N* by means of a particular order), Cantor was also able to discover that *Z* and *Q* have the same power as *N*. Having a specific difference in common (countability), they all belong to the same type (Lat. *genus*), which he defined as the "transfinite of minimal order": the fundamental transfinite order of the infinite with the power of a countable set. But since Cantor did not know the metaphysical Thomistic principle regarding the real difference between essence and existence, and persisting in his Parmenidean prejudice with respect to the existence of any logically non inconsistent entity (essence = existence), he did not recognize that if he had re-read his discovery in the light of the essence-existence difference, he would have found the way to obtain the generalized and universal "principle of limitation of the extension of a set" that he sought in vain throughout his life and which would have solved every antinomy. Thus, he continued to interpret the capacity for incremental increase of a transfinite set as a purely logical contingency. He also continued to search "elsewhere" for those elements that are not actually defined inside the transfinite continuum and to consider them as formally existing within the interior of the Absolute collection and thereby failing in the attempt to found, on the transfinite, the *reductio ad unum* that he had so long and unsuccessfully looked for. From here arose the final inconsistency of his justification of the notion of set and, in particular, of the notion of number based on set theory, which opened up the road to future axiomatizations.

3. The Power of the Continuum: From the Antinomy of the Power Set to the Axiom of the Power Set. Another very important consequence of the Cantorian theory is the fact that real numbers, placed by Dedekind into a biunivocal correspondence with the points of the "continuum," and therefore the continuum itself, exceed the power of a countable set. In fact real numbers are defined by Dedekind as limits of a sequence of rational numbers, limits that are "outside" each of these sequences. Cantor then posed the following problem. The power of the continuum is greater than the power of a countable set (called *Aleph*-0), but is the power of the continuum the one immediately greater than that of a countable set? Since Cantor did not succeed in constructing transfinite cardinals that were included between the power of the countable and the power of the continuum, he formulated the conjecture of the *continuum hypothesis* (Continuum Hypothesis, abbreviated CH), i.e., “the continuum has the power immediately greater than that of the countable.”

Immediately related to the notion of the power set understood as “the set of all the subsets of a given set” is the “antinomy of the power set”; it is precisely a contradiction arising when considering a set of maximum power, or a “universal set.” If we observe the construction of the power set of a set *A*, we see that it includes also *A* itself as one of its subsets. It follows that “any set is a [proper] subset of its power set; therefore the set of all sets, or universal set *U*, may not exist,” because it should be a proper subset of its power set, and it would therefore no longer be the universal set, in contradiction with its definition.

As we have seen, the demonstration of the antinomy of the power set was functional for the Neoplatonic notion of “Absolute collection” and therefore for the notion of transcendence as pure “indescribability.” Within Cantor’s thought, this notion was strongly conditioned by the Augustinian theology relating to number, which placed the actually infinite totality of the numbers as fully known within the mind of God (cf. Augustine, *De civitate Dei*, XII, 18; cf. Hallett 1984, pp. 35-37).

Then, later on, the antinomy of Burali-Forti was discovered. He showed that not only is the idea of a universal set founded on cardinal numbers inconsistent but also that the idea of “absolute ordering” leads to a contradiction. The latter depends on the original properties of the transfinite ordinals. In fact, while the properties of the finite ordinal numbers coincide with those of the finite cardinals, this is not so for the infinite ordinals. This antinomy placed into crisis the Cantorian idea of “absolute” as a maximal set of all the transfinite orderings. In fact, if this maximal ordinal set was to exist, its limiting element should belong (as it is the maximal set) and at the same time should not belong (as it is the limiting element) to the ordinal set that it orders.

V. Concluding Observations

The concept of infinity, in conclusion, is a notion with more than just a passing significance. Departing from the first thinkers who generally identified it with the “indeterminate” (Anaximander, Parmenides), or the “limitless,” essentially acquiring the negative value of imperfection, the concept then developed (thanks to the comprehension of the different modes of significance of the transcendental notion of “entity” [*analogia enti*]) until it assumed a positive value of full and perfect actuality, deprived of every limitation (Plato and Aristotle). This course of development of the notion with its diverse meanings was possible thanks to the concurrent contributions and cooperation of mathematics, metaphysics, and theology. On the mathematical side the need arose to explain multiplicity from a numerical point of view: Not only the concept of “one” exists, but also the concept of “two,” as the Pythagoreans observed. From this dyad, the construction of the sequence of the natural numbers became possible, and then the sequence of the integers and of the rational numbers, understood as successors of “one” (as, in modern times, can be found in the axiomatization of the Italian mathematician Giuseppe Peano, who based his concepts on the ideas of “one” and of “successor”), and finally the set of the irrational numbers, which are defined as limits to which certain sequences of rational numbers tend (Dedekind). On the metaphysical side, the need arose to explain the multiplicity of being (and therefore the nature of motion and becoming), which the Parmenidean perspective considers purely an appearance, and also the need to resolve the contradiction of a totally undifferentiated entity (Melissus).

This multiplicity, founded once for the entity as such in physics-metaphysics (present as early as Democritus), and for number (and later for the sets) in mathematics, could present itself, at least conceptually, as finite or as infinite, as capable or incapable of incremental increase. Aristotle, with his doctrine of potency and act, succeeded in explaining the differentiated multiplicity of the entity and in conceiving infinity as potential, and therefore as actually finite but capable of incremental increase to infinity, or actually infinite and not capable of further incremental increase. Thomas Aquinas and Cantor distinguished two types of actual infinity: relative (*secundum quid* for Aquinas and "transfinite" for Cantor) and absolute. In fact, an infinite object, when actually considered as a unique "thing," can be "relatively" infinite when it does not include in its interior something that delimits it, as happens for the sequence of the natural numbers; it can be "absolutely" infinite when an infinity does not contain some limitations regarding either its interior or exterior, being therefore perfectly actualized, as can only occur in God. And it is only thanks to the analogy of being (Lat. *analogia entis*) that we can conceive this infinite actuality absolutely in act—avoiding the Parmenidean contradiction—as placed at a level of being that is "beyond every other type" of entity, beyond every distinction of genera and of difference (transcendence). On this level metaphysics encounters theology: The absolute actual infinity of the metaphysical God is recognized in the revealed God of theology. The former is a glimpse, caught by natural reason, that reveals the existence of God and His main attributes, among which is infinity; the latter is received by us through Revelation, which is the initiative of God Himself that discloses to human reason both what we can know of Him through reason (albeit with difficulty and with the capacity for error) and also that which, left to our own mental powers, we could never even imagine.

Thus what is relevant from a logical point of view, both for mathematics and for metaphysics or theology, is the following theoretical result, which is part of the common foundations of these three disciplines: To consider and speak about actual infinity, and even to consider many kinds of actual infinities, some of which are relative and only one of which is Absolute, is not contradictory.

In conclusion, I think it can be said we have learned today that, for a particular science, the rigor of an advanced formalism, which only a constructive approach can guarantee, can be used only with a subset of the proper objects of that science. To be precise, we have learned that we have to limit the extension of the sets we want to construct, that is, of those sets the existence of which can be demonstrated, and not only supposed, by means of some specific axiom. This is the challenge that presents itself today: Must this subset of all the objects we construct be fixed or can it vary, thereby varying the axioms of the related formal system? In other words, need we suppose that *all* the objects inside the universal collection are *actually* existing, or may we consider as *actually* existing, time after time, only different *subsets* of them, and all the others to be only *virtually* existing? Here, by "virtually existing" we mean "called to actually exist" by means of a suitable demonstrative procedure — and therefore constructively — every time a logical necessity arises for it. This is the suggestion that the Thomistic vision seems able to offer to the inquiry that challenges those scholars who are dealing today with the theory of foundations; they have in fact arrived at the threshold of metaphysics and are confronting the same problems that occupied the greatest minds of ancient times. This endeavor could lead to useful and mature fruits for mathematics, metaphysics, and theology. Such fruits would concern a theory offering common foundations for the diverse disciplines and constitute the indispensable premise for a synthesis that restores knowledge to its equilibrium and unity.

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